Similar statements for sine and cosine series for functions on $[0, L]$

__Example:__ Consider $f(x) = 100 \quad 0 \leq x \leq L$

Cosine series for f

$$A_0 = \frac{1}{L} \int_0^L 100 \cdot dx = 100$$

$$A_n = \frac{2}{L} \int_0^L \cos \frac{n\pi}{L} x \cdot 100 \, dx$$

$$= \frac{2}{L} \cdot \frac{n\pi}{L} \sin \frac{n\pi}{L} x \Big|_0^L = 0$$

$$\Rightarrow F(f)(x) = 100 = f(x) \quad \forall x$$

$\boxed{\text{sine series}}$

done before, book 2.3.7

given by $\displaystyle\sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$

where $B_n = \begin{cases} 0 & n \text{ even} \\ \frac{400}{n\pi} & n \text{ odd.} \end{cases}$

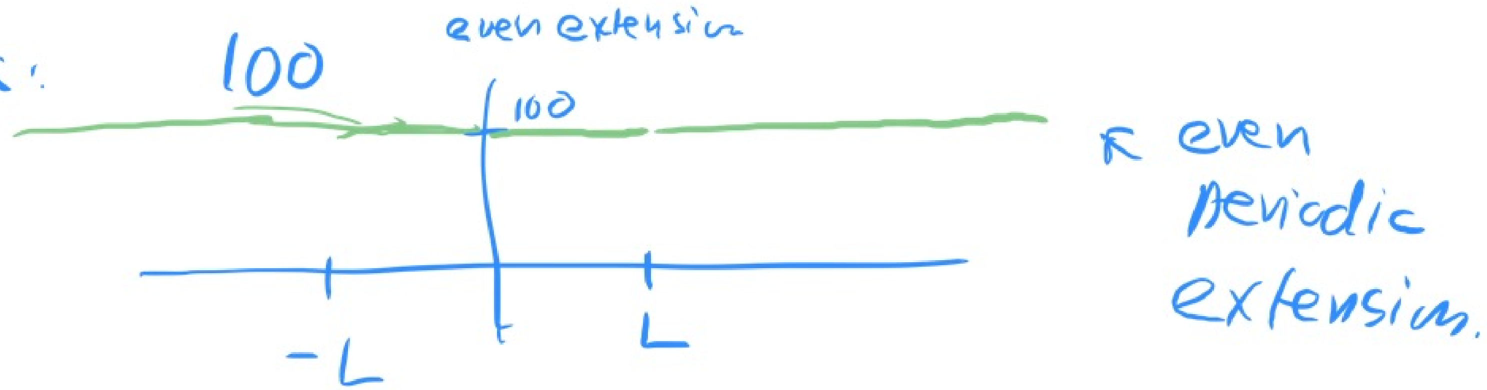
Fourier's theorem

$$\Rightarrow \quad 100 = \sum B_n \sin \frac{n\pi}{L} x$$

$$= \underbrace{\frac{400}{\pi} \sin \frac{\pi}{L} x}_{n=1} + \underbrace{\frac{400}{3\pi} \sin \frac{3\pi}{L} x}_{n=3} + \underbrace{\dots}_{n=5 \dots}$$

$$\boxed{100 = \frac{400}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin \frac{(2k+1)\pi}{L} x \quad \text{for all } x \; 0 < x < L}$$

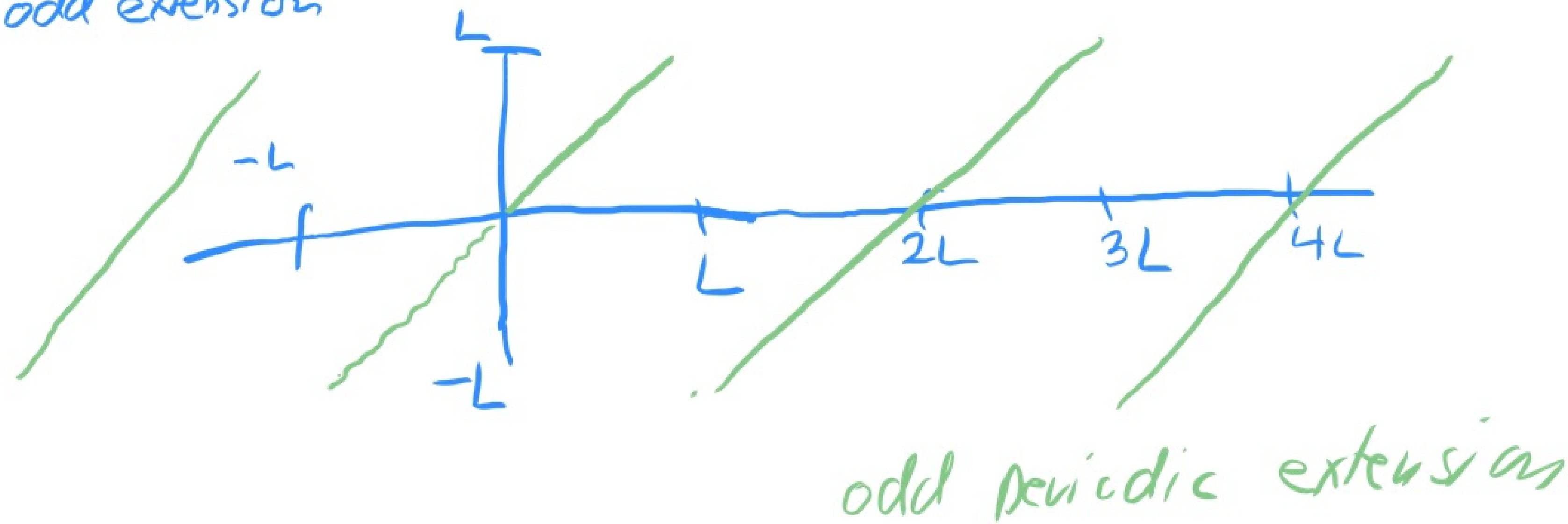
Cosine series:



Another example:

$f(x) = x, \qquad 0 \leq x \leq L$

odd extension



Question: Given $f : [0, L] \to \mathbb{R}$

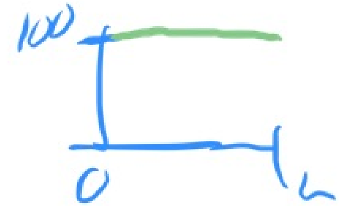
how are its sine and cosine series related?

Answer: extend sine and cosine series to $[-L, L]$

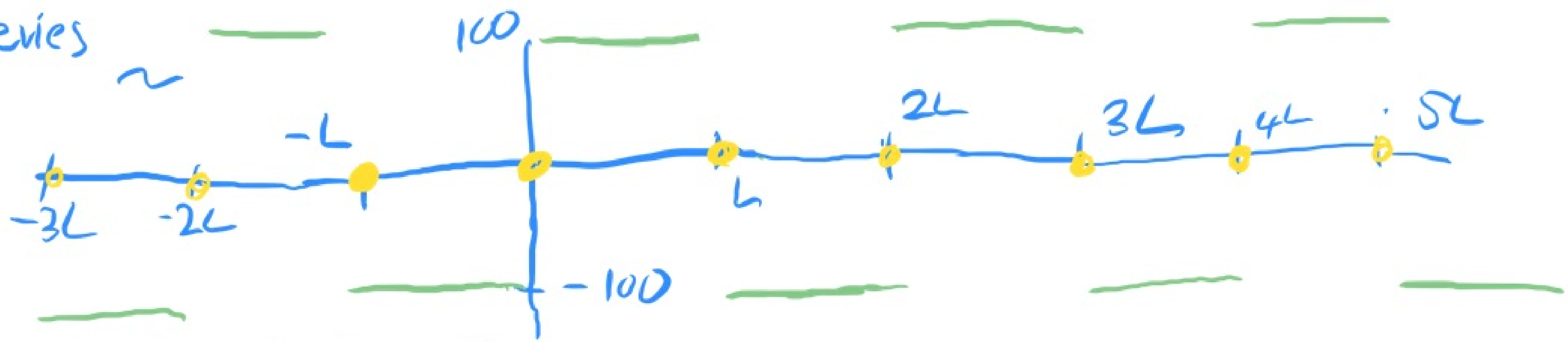
then: sine series $\sim$ odd extension of f to $[-L, L]$

cosine series $\sim$ even extension.

for our example: f



sine series $\sim$



= odd periodic extension of f

Calculate Fourier coef. of sine series:

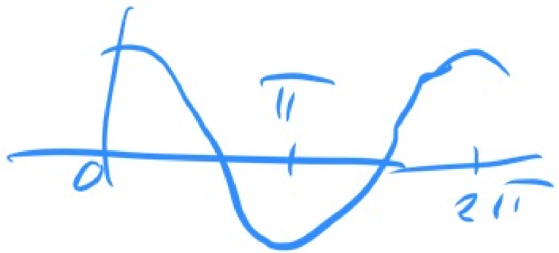
$$B_n = \frac{2}{L}\int_0^L \underbrace{x}_{u}\ \underbrace{\sin\frac{n\pi}{L}x}_{v'}\,dx$$

$$= \frac{2}{L}\left[-\frac{L}{n\pi}x\cos\frac{n\pi}{L}\Big|_0^L + \int_0^L +\frac{L}{n\pi}\cos\frac{n\pi}{L}x\,dx\right]$$

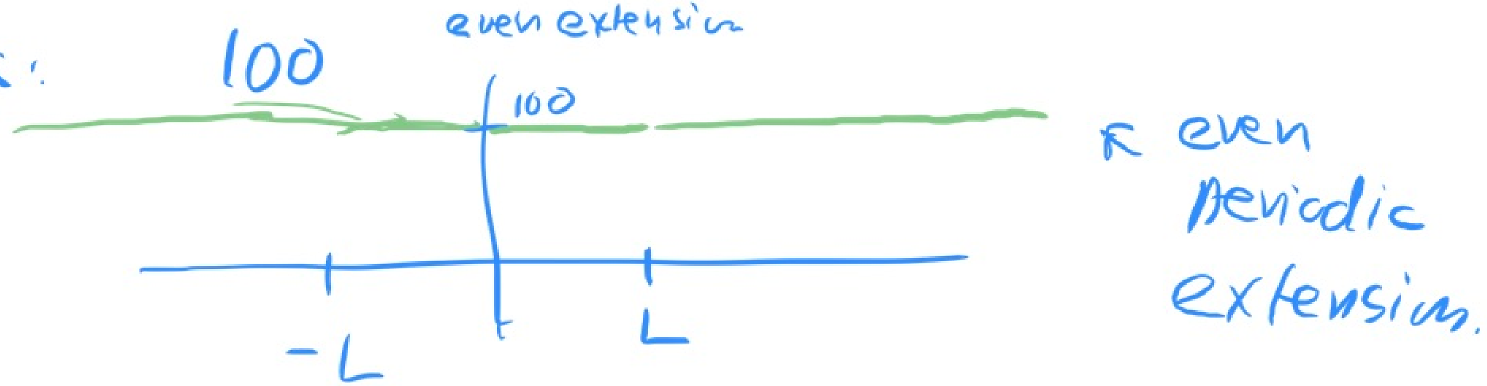
$$= -\frac{2L}{n\pi}\cos n\pi - 0 \qquad + \text{const.} \qquad \underbrace{\underbrace{\sin\frac{n\pi}{L}x\Big|_0^L}_{\sin n\pi - \sin 0}}_{=0}$$

$$= \boxed{(-1)^{n+1}\frac{2L}{n\pi} = B_n}$$

because $\cos n\pi = (-1)^n$



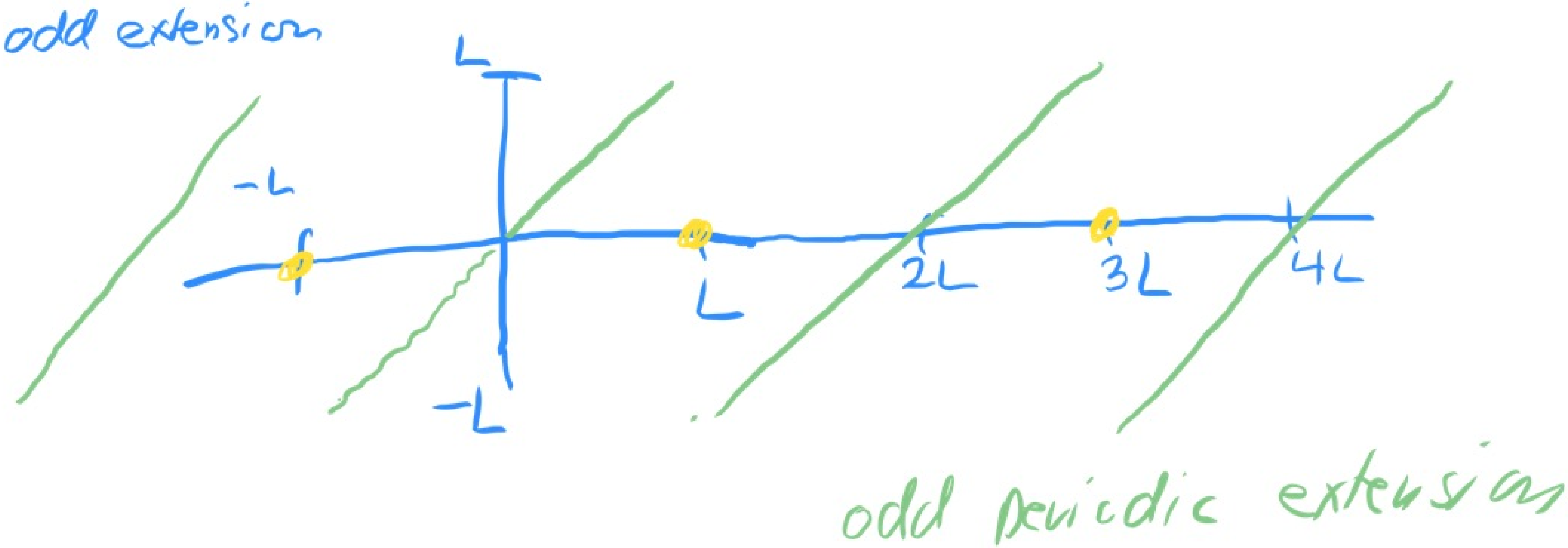
Cosine series:



Another example:

$$f(x) = x, \qquad 0 \leq x \leq L$$

odd extension



Fourier's Theorem

$$x = \sum B_n \sin \frac{n\pi}{L} x$$

$$= \sum \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{L} x$$

$$0 < x < L$$

$x = 0$?

get 0 on both sides.

can also be seen at periodic odd extension

no jumping point!

cosine series for $f(x) = x$

$$\sum A_n \cos \frac{n\pi}{L} x \qquad 0 \le x \le L$$

$$\Rightarrow \quad A_0 = \frac{1}{L} \int_0^L x \, dx = \frac{1}{L} \left. \frac{x^2}{2} \right|_0^L = \frac{L}{2}$$

$$A_n = \frac{2}{L} \int_0^L x \cos \frac{n\pi}{L} x \, dx$$

$=$ integration by parts ....

$$= \frac{2L}{(n\pi)^2} \big(\underbrace{\cos n\pi}_{=(-1)^n} - 1 \big) = \begin{cases} 0 & n \text{ even} \\ -\dfrac{4L}{(n\pi)^2} & n \text{ odd} \end{cases}$$

$\Rightarrow$ cosine series for $f(x) = x$ given by

$$\frac{L}{2} - \frac{4L}{\pi^2}\cos\frac{\pi}{L}x - \frac{4L}{9\pi^2}\cos\frac{3\pi}{L}x - \underbrace{\ldots}_{n=5}\ \ldots$$

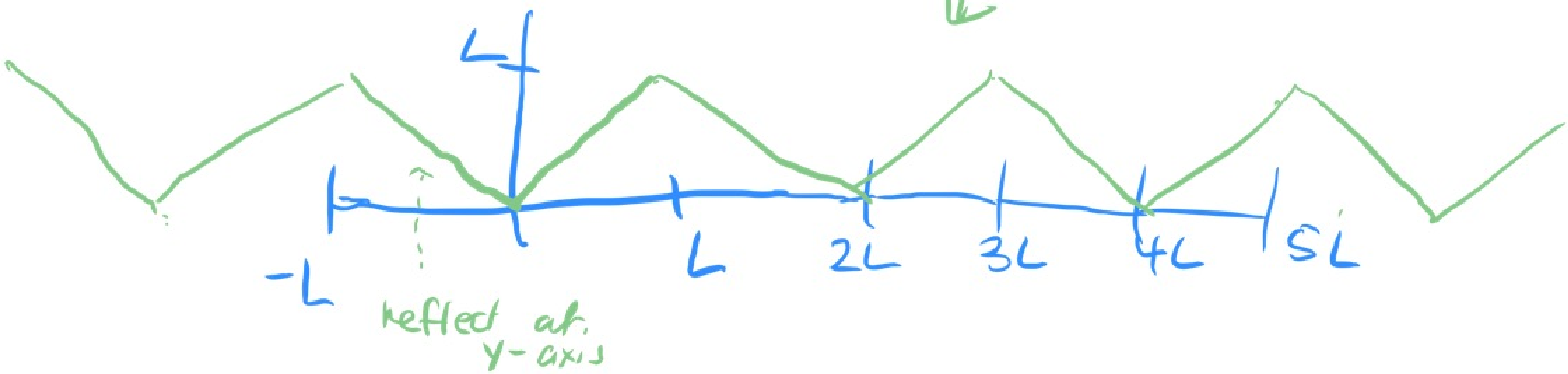
$$= \frac{L}{2} - \sum_{k=0}^{\infty} \frac{4L}{(2k+1)^2\pi^2}\cos\frac{(2k+1)\pi}{L}x$$

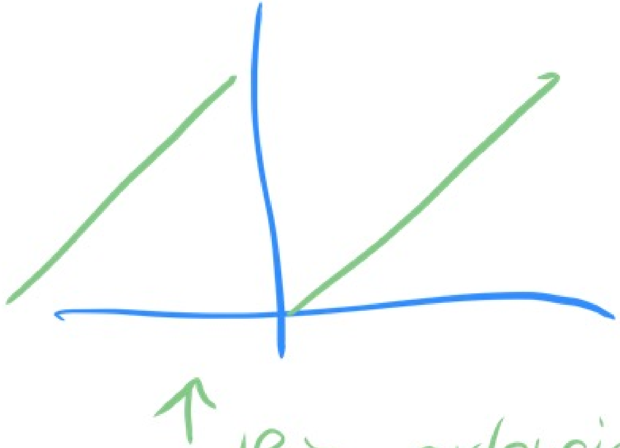
$$= x \quad \text{if no jumping point} \quad 0 \leq x \leq L$$

Fourier's Theorem

even extension:

periodic even extension = graph of cosine series

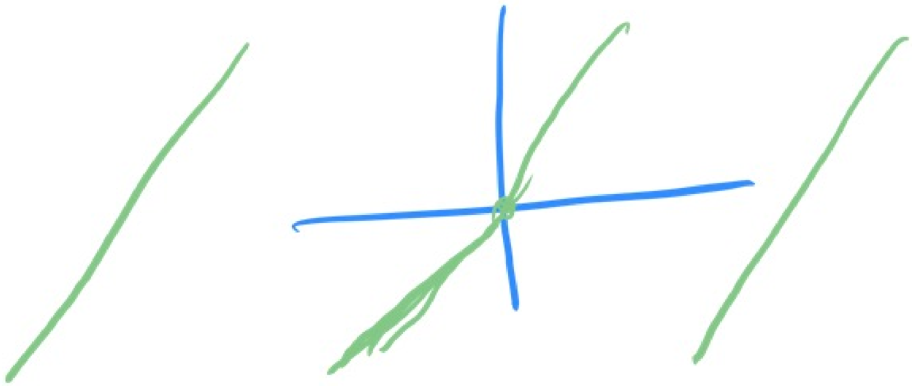




this extension is <u>not</u> even

<u>Remark:</u>
odd periodic extension of $f: [0,\pi) \to \mathbb{R}$

= graph of sine series of f.



Odd and Even Functions

f even: $\qquad f(-x) = f(x)$

f odd $\qquad f(-x) = -f(x)$

Theorem $\quad f: [-L, L] \to \mathbb{R}$

$\Rightarrow$ (a) can write f as

$$f = f_{ev} + f_{od}$$

where $f_{ev}(x) = \frac{1}{2}(f(x) + f(-x))$ even fct.

$f_{od}(x) = \frac{1}{2}(f(x) - f(-x))$ odd fct.

(b) Fourier Series of f

= cosine series of $f_{ev}|_{[0,L]}$ + sine series of $f_{odd}|_{[0,L]}$

For statement (b):

calculate coeff. a_n of Fourier series of f, $n>0$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$= \frac{1}{L}\int_{-L}^{L}\left(f_{ev}(x)+f_{od}(x)\right)\cos\frac{n\pi}{L}x\,dx$$

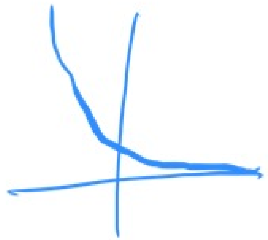
$$= \frac{1}{L}\int_{-L}^{L}\underbrace{f_{ev}(x)\cos\frac{n\pi}{L}x}_{\text{even}}\,dx + \underbrace{\frac{1}{L}\int_{-L}^{L}\underbrace{f_{od}\underbrace{\cos\frac{n\pi}{L}x}_{ev}}_{\text{odd}}\,dx}_{=0}$$

$$= \frac{1}{L}\,2\int_{0}^{L} f_{ev}(x)\cos\frac{n\pi}{L}x\,dx$$

$= A_n =$ coefficient of cosine series of $f_{ev}|_{[0,L]}$

Example: $f(x) = \frac{1}{1+x}$

$[-L, L] \to \mathbb{R}$



$$\Rightarrow f_{ev} = \frac{1}{2}\left(f(x) + f(-x)\right)$$

$$= \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)$$

$$= \frac{1}{2} \, \frac{1-x+1+x}{1-x^2} = \boxed{\frac{1}{1-x^2} = f_{ev}}$$

$$f_{odd} = \frac{1}{2}\left(\frac{1}{1+x} - \frac{1}{1-x}\right) = \dots \boxed{\frac{-x}{1-x^2} = f_{od}}$$

check $f_{ev} + f_{od} = \frac{1-x}{1-x^2} = \frac{1}{1+x}$ ✓

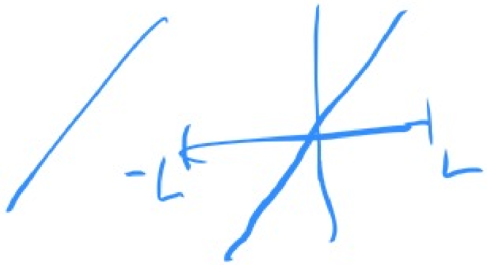
show same way:

b_n = coeff. B_n of sine series for $f_{odd}|_{[0,L]}$

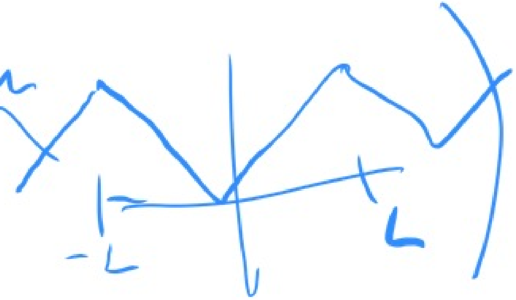
Let now $f: [0,L] \to \mathbb{R}$

have odd and even $\underbrace{\text{extensions}}_{\text{periodic}}$ of f

(recall: $f(x) = x$ odd:



even



<u>Theorem</u> Let $f: [0,L] \to \mathbb{R}$ be continuous

(a) Its <u>even</u> periodic extension is also continuous

and $f(x) = A_0 + \sum A_n \cos \frac{n\pi}{L} x$ cosine series.

for all x.

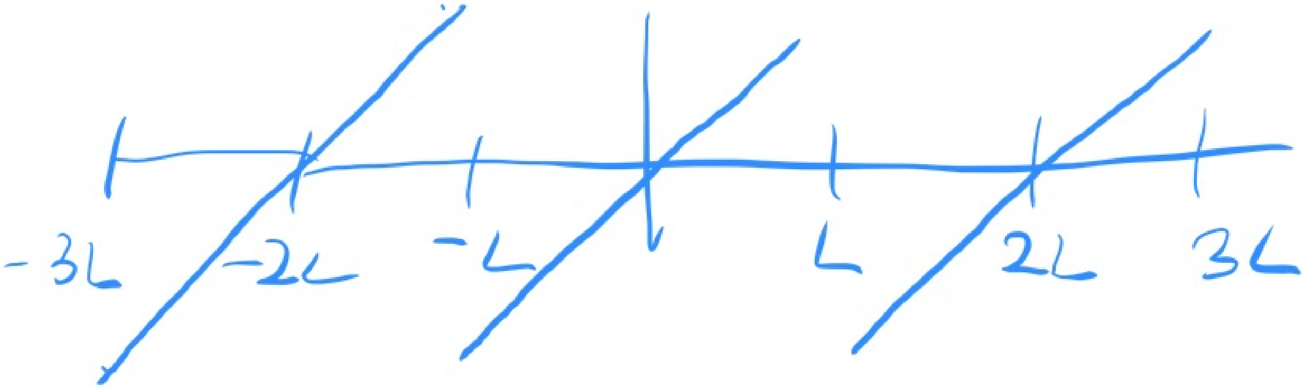
(b) Its <u>odd</u> periodic extension is continuous <u>only</u> if

$f(0) = 0 = f(L)$

If $f(0) \neq 0$ = value of its sine series

$f(L) \neq 0$ = value " " "

odd periodic extension



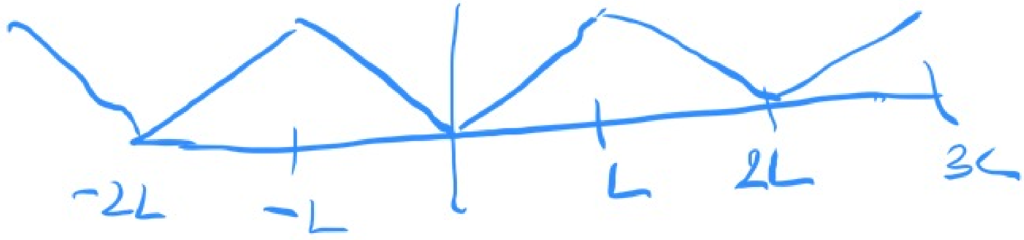
calculated sine series

$$x \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2L}{n\pi} \underbrace{\sin \frac{n\pi}{L} x}_{\sin n\pi = 0 \text{ for } x = L}$$

$f(L) = L \neq$ value of sine series for $x = L$

For our example $f(x) = x$

(a) periodic even extension



have calculated cosine series

theorem:

$$x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{L} x$$

$$= \frac{L}{2} - \sum_{n \text{ odd}} \frac{4L}{n^2\pi^2} \cos \frac{n\pi}{L} x$$

calculated last class

for all $x \in \mathbb{R}$!

3.4 Term by Term Differentiation of Fourier Series

Question: assume have calculated Fourier series for some function f

$\Rightarrow$ do we get Fourier series for $f' = \frac{df}{dx}$ by just differentiating the Fourier series ?

⚠ Does not always work.

Example: $f(x) = x$

which has sine series

$$x \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2L}{n\pi} \sin \frac{n\pi}{L} x$$

$\frac{d}{dx} x = 1$ ✓

differentiate sine series :
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\cancel{L}}{\cancel{n\pi}} \left(\cos \frac{n\pi}{L} x \right) \cdot \frac{\cancel{n\pi}}{\cancel{L}}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \, 2 \cos \frac{n\pi}{L} x$$

Observe: (a) This is not the cosine series of $\frac{d}{dx} x = 1$!

(cosine series $= 1$)

(b) This series does NOT converge

(recall: a series $\sum_{n=1}^{\infty} a_n$ can only converge if $a_n \to 0$ for $n \to \infty$

$\cos \frac{n\pi}{L} x \not\to 0$ for $n \to \infty$

Useful

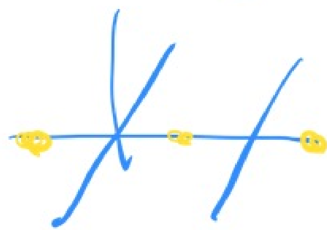
Theorem $f: [-L, L] \to \mathbb{R}$ piecewise smooth

If

- Fourier series for f is continuous
- $f'(x)$ is piecewise smooth

$\Rightarrow$ Fourier series of $f'(x)$ can be obtained from the one of f by differentiating term by term

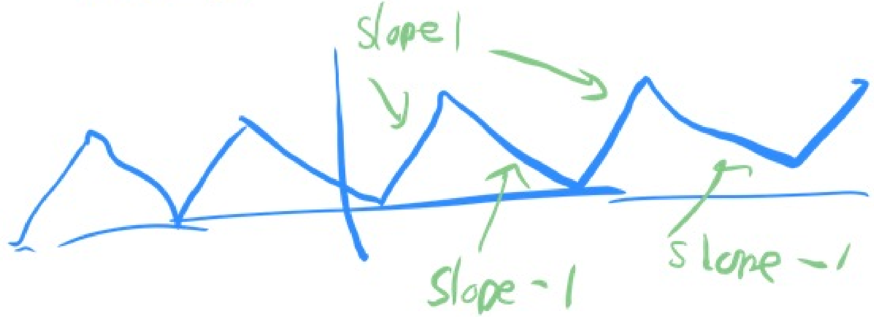
Remark: ① In our previous example, sine series for odd extension of $f(x)$ Fourier series (= sine series) not continuous



jumps

Conditions of theorem not satisfied

(2) Even extension for $f(x)=x$ continuous



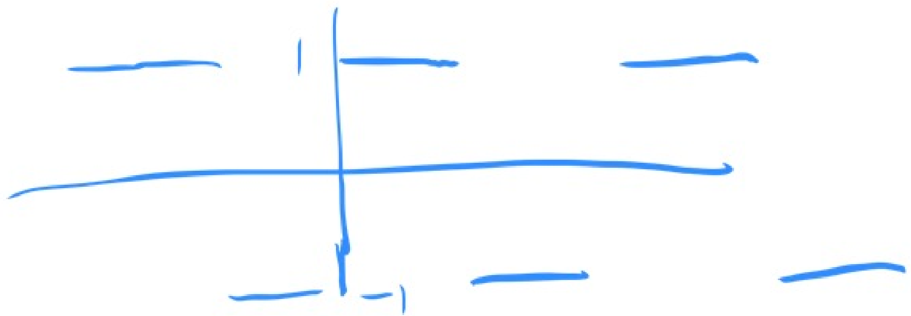
by previous theorem:

$f(x)$ coincides with its cosine series

contin. contin.

first cond. satisfied

$f'(x)$



$f'(x)$ piecewise smooth

$\Rightarrow$ can apply theorem.

$\Rightarrow$ $f'(x)$ has sine series given by

$$\frac{d}{dx}\left(\underset{\substack{\downarrow\\0}}{\frac{L}{2}} - \sum_{n \text{ odd}} \frac{4L}{n^2\pi^2}\cos\frac{n\pi}{L}x\right)$$

$$= -\sum_{n \text{ odd}} \frac{4\cancel{L}}{n^{\cancel{2}}\pi^{\cancel{2}}}\left(-\sin\frac{n\pi}{L}x\right)\frac{n\cancel{\pi}}{\cancel{L}}$$

$$= \boxed{\sum_{n \text{ odd}} \frac{4}{n\pi}\sin\frac{n\pi}{L}x}$$

= Fourier series for